

# Semi-infinite $q$ -wedge construction of the level 2 Fock Space of $U_q(\hat{\mathfrak{sl}}_2)$ \*

Jens-Ulrik Holger Petersen <sup>†</sup>

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## Abstract

In this proceedings a particular example from [KMPY] is presented: the construction of the level 2 Fock space of  $U_q(\hat{\mathfrak{sl}}_2)$ . The generating ideal of the wedge relations is given and the wedge space defined. Normal ordering of wedges is defined in terms of the energy function. Normally ordered wedges form a base of the wedge space.

The  $q$ -deformed Fock space is defined as the space of semi-infinite wedges with a finite number of vectors in the wedge product differing from a ground state sequence and endowed with a separated  $q$ -adic topology. Normally ordered wedges form a base of the Fock space. The action of  $U_q(\hat{\mathfrak{sl}}_2)$  on the Fock space converges in the  $q$ -adic topology. On the Fock space the action of bosons, which commute with the  $U_q(\hat{\mathfrak{sl}}_2)$ -action, also converges in the  $q$ -adic topology. Hence follows the decomposition of the Fock space into irreducible  $U_q(\hat{\mathfrak{sl}}_2)$ -modules.

## 1 Introduction

The classical semi-infinite wedge construction [DJKM] originates from the study of the representation theory of affine Lie algebras and the soliton theory of integrable hierarchies during the 80's (for a review see for example [KR]). In the last couple of years the subject has been revived by the consideration of its  $q$ -deformation in the context of the representation theory of quantum affine algebras.

In [S, KMS] the semi-infinite wedge space construction of the level 1 Fock space of  $U_q(\hat{\mathfrak{sl}}_n)$  and its decomposition were given. In [KMPY] under certain assumptions a general scheme for the wedge construction of  $q$ -deformed Fock spaces using the theory of perfect crystals was presented. Let  $U_q(\mathfrak{g})$  be a

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\*This proceedings is based on the paper "Perfect crystals and  $q$ -Fock spaces" [KMPY] by Masaki Kashiwara, Tetsuji Miwa, Chung Ming Yung and the speaker.

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quantum affine algebra. Let  $V$  be a finite dimensional  $U_q(\mathfrak{g}')$ -module with a perfect crystal base [KMN] of level  $k$ . Let  $V_{\text{aff}}$  denote its affinization. In [KMPY] the wedge space  $\bigwedge^r V_{\text{aff}}$  ( $r \in \mathbb{N}$ ) was constructed using the  $U_q(\mathfrak{g})$ -action and the following  $U_q(\mathfrak{g})$ -linear map was given

$$\bigwedge^r V_{\text{aff}} \otimes \bigwedge^s V_{\text{aff}} \longrightarrow \bigwedge^{r+s} V_{\text{aff}}.$$

Normal ordering of wedges was defined and it was proven that normally ordered wedges form a base of  $\bigwedge^r V_{\text{aff}}$ . The level  $k$  Fock space  $\mathcal{F}_m$  ( $m \in \mathbb{Z}$ ) was constructed and the corresponding  $U_q(\mathfrak{g})$ -linear map

$$\bigwedge^r V_{\text{aff}} \otimes \mathcal{F}_m \longrightarrow \mathcal{F}_{m-r}$$

was defined. The decomposition of the Fock space was given.

In [KMPY] examples of the theory for level 1  $A_n^{(1)}$  ([KMS]),  $B_n^{(1)}$ ,  $D_n^{(1)}$ ,  $A_{2n}^{(2)}$ ,  $A_{2n-1}^{(2)}$ ,  $D_{n+1}^{(2)}$  and level  $k$   $A_1^{(1)}$  were also given.

In this talk I summarize some of the results of [KMPY], describing the case of level 2  $U_q(\hat{\mathfrak{sl}}_2)$  in detail. This fairly simple example is sufficient to illustrate the differences between the level 1  $U_q(\hat{\mathfrak{sl}}_n)$  case ([KMS]) and other cases ([KMPY]): in particular the need to endow the Fock space with a separated  $q$ -adic topology. For fuller details and proofs, please see [KMPY].

## 1.1 Thanks

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## 2 Preliminaries

Let  $\mathfrak{g}$  be an affine Lie algebra with associated weight lattice  $P := \sum_{i \in I} \mathbb{Z} \Lambda_i \oplus \mathbb{Z} \delta$  and  $\mathfrak{g}'$  its derived Lie subalgebra with associated weight lattice  $P_{\text{cl}} := \sum_{i \in I} \mathbb{Z} \Lambda_i^{\text{cl}}$ . Define  $h_i \in P^*$  by  $\langle h_i, \delta \rangle = 0$  and  $\langle h_i, \Lambda_j \rangle = \delta_{i,j}$  ( $i, j \in I$ ). Similarly define  $h_i \in P_{\text{cl}}^*$  by  $\langle h_i, \Lambda_j^{\text{cl}} \rangle = \delta_{i,j}$  ( $i, j \in I$ ). Let  $W$  be the Weyl group of  $\mathfrak{g}$ . Let  $c$  denote the canonical central element in  $\mathfrak{g}$  and  $\mathfrak{g}'$ . Define the projection  $\text{cl} : P \rightarrow P_{\text{cl}}$  by

$$\text{cl}\left(\sum_{i \in I} \omega_i \Lambda_i + \omega_\delta \delta\right) := \sum_{i \in I} \omega_i \Lambda_i^{\text{cl}} \quad (\omega_i, \omega_\delta \in \mathbb{Z}).$$

Define  $P_{\text{cl}}^+ := \{\lambda \in P_{\text{cl}} \mid \langle h_i, \lambda \rangle \geq 0\}$ .

Let  $U_q(\mathfrak{g})$  (respectively  $U_q(\mathfrak{g}')$ ) be the quantum universal enveloping algebra of  $\mathfrak{g}$  ( $\mathfrak{g}'$ ) over  $\mathbb{Q}(q)$  with generators  $e_i, f_i, q^h$  ( $i \in I$  and  $h \in P^* (P_{\text{cl}}^*)$ ). Write  $t_i$  for  $q_i^{h_i}$ .

The coproduct of  $U_q(\mathfrak{g})$  ( $U_q(\mathfrak{g}')$ ) is taken to be  $\Delta = \bar{\Delta}_+$ :

$$\Delta : \begin{cases} q^h \mapsto q^h \otimes q^h & (h \in P^* (P_{\text{cl}}^*)) \\ e_i \mapsto e_i \otimes 1 + t_i^{-1} \otimes e_i & (i \in I) \\ f_i \mapsto f_i \otimes t_i + 1 \otimes f_i & (i \in I) \end{cases}.$$

In this talk I take  $\mathfrak{g} = \hat{\mathfrak{sl}}_2$ , so  $I = \{0, 1\}$  and  $c = h_0 + h_1$ .

## 2.1 Perfect crystal base

I recall briefly some facts from the theory of crystals bases (see [K] for the definitions). Recall that a *crystal* is a set  $B$  with maps  $\tilde{e}_i, \tilde{f}_i : B \sqcup \{0\} \rightarrow B \sqcup \{0\}$  ( $i \in I$ ), such that

- (i)  $\tilde{e}_i 0 = 0 = \tilde{f}_i 0$ ,
- (ii) there exists  $n \in \mathbb{Z}_{>0}$  such that  $\tilde{e}_i^n b = 0 = \tilde{f}_i^n b$  ( $\forall b \in B, i \in I$ ),
- (iii)  $b' = \tilde{f}_i b \iff b = \tilde{e}_i b'$  ( $b, b' \in B, i \in I$ )

Let  $V$  be a finite dimensional  $U_q(\mathfrak{g}')$  module with crystal base  $(L, B)$ . We take the  $U_q(\hat{\mathfrak{sl}}_2)$ -module to be  $V := \bigoplus_{j \in J := [0, 2]} \mathbb{Q}(q)v_j$  with crystal  $B = \bigsqcup_{j \in J} \{b_j\}$  and crystal graph

$$b_0 \xrightleftharpoons[0]{1} b_1 \xrightleftharpoons[0]{1} b_2.$$

In the crystal graph an arrow  $b \xrightarrow{i} b' \iff b' = \tilde{f}_i b$ .

Let  $A := \{f \in \mathbb{Q}(q) \mid f \text{ has no pole at } q = 0\}$ . The *crystal lattice*  $L = A \otimes_{\mathbb{Q}} \text{span}_{\mathbb{Q}}(B)$ .

Define maps  $\varepsilon_i, \varphi_i : B \rightarrow \mathbb{N}$  by

$$\begin{aligned} \varepsilon_i(b) &:= \max\{n \in \mathbb{N} \mid \tilde{e}_i^n b \neq 0\}, \\ \varphi_i(b) &:= \max\{n \in \mathbb{N} \mid \tilde{f}_i^n b \neq 0\}. \end{aligned}$$

and maps  $\varepsilon, \varphi : B \rightarrow P_{\text{cl}}$  by

$$\varepsilon(b) := \sum_{i \in I} \varepsilon_i(b) \Lambda_i^{\text{cl}}, \quad \varphi(b) := \sum_{i \in I} \varphi_i(b) \Lambda_i^{\text{cl}}.$$

In our example

$$\begin{aligned} \varepsilon(b_j) &= (2 - j) \Lambda_0^{\text{cl}} + j \Lambda_1^{\text{cl}} \\ \varphi(b_j) &= j \Lambda_0^{\text{cl}} + (2 - j) \Lambda_1^{\text{cl}} \end{aligned} \quad (j \in J). \quad (2.1.1)$$

The weight of  $b \in B$  is given by  $\text{wt}(b) = \varphi(b) - \varepsilon(b)$ :

$$\text{wt}(b_j) = 2(1-j)(\Lambda_1^{\text{cl}} - \Lambda_0^{\text{cl}}).$$

We take  $\{v_j = G(b_j)\}_{j \in J}$  to be a *lower global base*  $[K]$  of  $V$ . For our example, we have

$$\begin{aligned} e_i G(b) &= [\varphi_i(b) + 1]G(\tilde{e}_i b), \\ f_i G(b) &= [\varepsilon_i(b) + 1]G(\tilde{f}_i b), \\ q^h G(b) &= q^{\langle h, \text{wt}(b) \rangle} G(b). \end{aligned}$$

The use of a lower global base is essential to our construction: see the remarks after Lemma 3.1 and Theorem 4.1.

Let  $B = \bigsqcup_{\lambda \in P_{\text{cl}}} B_\lambda$  be the weight decomposition.

Let  $B_1, B_2$  be crystals. The tensor product  $B_1 \otimes B_2$  (corresponding to the coproduct  $\Delta$ ) is defined to be the set  $B_1 \times B_2$  with the action of the Kashiwara operators  $\tilde{e}_i, \tilde{f}_i$  given by

$$\begin{aligned} \tilde{e}_i(b \otimes b') &= \begin{cases} \tilde{e}_i b \otimes b' & \text{if } \varepsilon_i(b) > \varphi_i(b') \\ b \otimes \tilde{e}_i b' & \text{if } \varepsilon_i(b) \leq \varphi_i(b') \end{cases}, \\ \tilde{f}_i(b \otimes b') &= \begin{cases} \tilde{f}_i b \otimes b' & \text{if } \varepsilon_i(b) \geq \varphi_i(b') \\ b \otimes \tilde{f}_i b' & \text{if } \varepsilon_i(b) < \varphi_i(b') \end{cases}. \end{aligned}$$

The crystal base  $(L, B)$  is perfect of level  $k = 2$ , which means that it satisfies the following conditions:

- (i) There exists a weight  $\lambda^\circ \in \text{wt}(B)$  such that  $\text{wt}(B)$  is contained in the convex hull of  $W\lambda^\circ$  and that  $|B_{w\lambda^\circ}| = 1$  ( $w \in W$ ). The elements of  $B_{w\lambda^\circ}$  and their weights  $w\lambda^\circ$  are called *extremal*.
- (ii)  $B \otimes B$  is connected as a crystal graph.
- (iii)  $k = \max\{n \in \mathbb{Z}_{>0} \mid \langle c, \varepsilon(b) \rangle \geq n \ (\forall b \in B)\}$ .
- (iv)  $\varepsilon, \varphi : \{b \in B \mid \langle c, \varepsilon(b) \rangle = k\} \rightarrow (P_{\text{cl}}^+)_k := \{\lambda \in P_{\text{cl}}^+ \mid \langle c, \lambda \rangle = k\}$  are bijective.

Note that  $\langle c, \varphi(b) - \varepsilon(b) \rangle = 0$ , since  $\langle c, \text{wt}(b) \rangle = 0$  ( $b \in B$ ).

In our example  $\lambda^\circ = \pm 2(\Lambda_1^{\text{cl}} - \Lambda_0^{\text{cl}})$ ,

$$\begin{array}{ccccc} b_0 \otimes b_0 & \xrightarrow{1} & b_0 \otimes b_1 & \xrightarrow{1} & b_0 \otimes b_2 \\ \uparrow 0 & & \uparrow 0 & & \downarrow 1 \\ B \otimes B = & b_1 \otimes b_0 & \xrightarrow{1} & b_1 \otimes b_1 & \xleftarrow{0} b_1 \otimes b_2 \\ \uparrow 0 & & \downarrow 1 & & \downarrow 1 \\ & b_2 \otimes b_0 & \xleftarrow{0} & b_2 \otimes b_1 & \xleftarrow{0} b_2 \otimes b_2 \end{array}$$

and (2.1.1) implies that  $\varepsilon, \varphi$  map bijectively.  $b_0, b_2$  are extremal.

## 2.2 Affinization

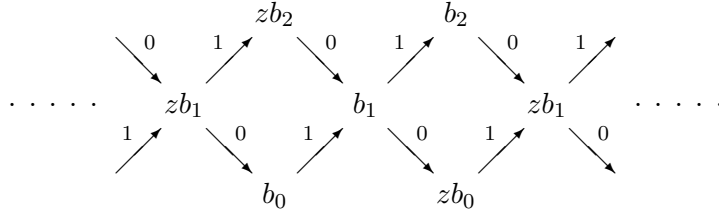
Let  $V_{\text{aff}} := V \otimes \mathbb{Q}[z, z^{-1}]$  be the  $U_q(\mathfrak{g})$ -module, which is the affinization of  $V$  such that

$$\begin{aligned} e_i(v \otimes \xi) &:= (e_i v) \otimes z^{\delta_{i,0}} \xi, \\ f_i(v \otimes \xi) &:= (f_i v) \otimes z^{-\delta_{i,0}} \xi, \\ \text{wt}(v_j \otimes z^a) &:= \text{wt}(v_j) + a\delta, \end{aligned} \quad (v \in V, \xi \in \mathbb{Q}[z, z^{-1}], j \in J, a \in \mathbb{Z}).$$

Let  $(L_{\text{aff}}, B_{\text{aff}})$  be the crystal base of  $V_{\text{aff}}$ .

Let  $z^a : V_{\text{aff}} \rightarrow V_{\text{aff}}$  ( $a \in \mathbb{Z}$ ) denote the  $U_q(\mathfrak{g}')$ -linear endomorphism  $v \otimes \xi \mapsto v \otimes z^a \xi$  ( $v \otimes \xi \in V_{\text{aff}}$ ). Then  $z^a v_j := z^a(v_j \otimes 1) \equiv v_j \otimes z^a$  ( $j \in J, a \in \mathbb{Z}$ ).

The crystal  $B_{\text{aff}} = \bigsqcup_{j \in J, a \in \mathbb{Z}} \{z^a b_j\}$ . The crystal graph of  $B_{\text{aff}}$  has the following structure.



The arrows of the crystal graph induce a partial ordering of  $B_{\text{aff}}$ .

Define the map  $l : B_{\text{aff}} \rightarrow \mathbb{Z}$  by

$$l(z^a b_j) := 2a - j.$$

Define the energy function ([KMN])  $H : B_{\text{aff}} \otimes B_{\text{aff}} \rightarrow \mathbb{Z}$  by

- (i)  $H(zb \otimes b') = H(b \otimes b') - 1$  and  $H(b \otimes zb') = H(b \otimes b') + 1$  ( $b, b' \in B_{\text{aff}}$ ),
- (ii)  $H$  is constant on every connected component of the crystal graph  $B_{\text{aff}} \otimes B_{\text{aff}}$ ,
- (iii)  $H(z^a b^\circ, z^a b^\circ) = 0$  for any extremal  $b^\circ \in B$  and  $a \in \mathbb{Z}$ .

For our example we have

$$H(b_i \otimes b_j) = \min\{i, 2 - j\} \quad (i, j \in J).$$

If  $H(b \otimes b') \leq 0$ , then  $l(b) \geq l(b')$ .

## 3 Wedge space

For an extremal  $b^\circ \in B$ , let  $v^\circ := G(b^\circ)$ . Define

$$N := U_q(\mathfrak{g})[z \otimes z, z^{-1} \otimes z^{-1}, z \otimes 1 + 1 \otimes z](v^\circ \otimes v^\circ).$$

It is not too difficult to prove that  $N$  is independent of the choice of extremal element  $b^\circ$ : indeed for any extremal  $b' \in B$ ,  $G(z^a b') \otimes G(z^a b') \in N$  ( $a \in \mathbb{Z}$ ).

For our example, define the following elements of  $N$ :

$$\begin{aligned}
C_{b_0, b_0} &:= v_0 \otimes v_0, \\
C_{b_0, b_1} &:= v_0 \otimes v_1 + q^2 v_1 \otimes v_0, \\
C_{b_0, b_2} &:= v_0 \otimes v_2 + q v_1 \otimes v_1 + q^4 v_2 \otimes v_0, \\
C_{b_1, b_2} &:= v_1 \otimes v_2 + q^2 v_2 \otimes v_1, \\
C_{b_2, b_2} &:= v_2 \otimes v_2, \\
C_{z b_2, b_1} &:= z v_2 \otimes v_1 + q^2 v_1 \otimes v_2, \\
C_{z^2 b_2, b_0} &:= z^2 v_2 \otimes v_0 + q z v_1 \otimes z v_1 + q^4 v_0 \otimes z^2 v_2, \\
C_{z b_1, b_0} &:= z v_1 \otimes v_0 + q^2 v_0 \otimes z v_1, \\
C_{z b_1, b_1} &:= z v_1 \otimes v_1 + q^2 v_1 \otimes z v_1 + q^2 [2](v_0 \otimes z v_2 + z v_2 \otimes v_0).
\end{aligned}$$

(These elements are constructed by the action of  $U_q(\hat{\mathfrak{sl}}_2)$  on  $v_1 \otimes v_1$ .)

**Lemma 3.1.** *Let  $\mathcal{B}_{i,j} := \{(b, b') \in B_{\text{aff}} \otimes B_{\text{aff}} \mid H(b \otimes b') > 0, l(b_j) \leq l(b) < l(z^{H(b_i \otimes b_j)} b_i), l(b_j) < l(b') \leq l(z^{H(b_i \otimes b_j)} b_i)\}$ . Each element  $C_{z^{H(b_i \otimes b_j)} b_i, b_j}$  has the form*

$$G(z^{H(b_i \otimes b_j)} b_i) \otimes G(b_j) - \sum_{(b, b') \in \mathcal{B}_{i,j}} a_{b, b'} G(b) \otimes G(b').$$

The coefficients  $a_{b, b'}$  lie in  $q\mathbb{Z}[q]$ .  $H(z^{H(b_i \otimes b_j)} b_i \otimes b_j) = 0$ .

The conditions in this Lemma are required for the construction of the Fock space. Note that this Lemma does not hold for the corresponding elements of  $N$  in an upper global base.

The vectors  $\{C_{z^{H(b_i \otimes b_j)} b_i, b_j}\}$  are linearly independent. We have

$$\sum_{i, j \in J} \mathbb{Q}(q)[z \otimes z, z^{-1} \otimes z^{-1}, z \otimes 1 + 1 \otimes z] C_{z^{H(b_i \otimes b_j)} b_i, b_j} = N.$$

Define the wedge product by

$$\bigwedge^2 V_{\text{aff}} := V_{\text{aff}} \otimes V_{\text{aff}} / N.$$

For  $v, v' \in V_{\text{aff}}$ , denote the image of  $v \otimes v' \in V_{\text{aff}}^{\otimes 2}$  in  $\bigwedge^2 V_{\text{aff}}$  by  $v \wedge v'$ .

We call a pair  $(b, b') \in B_{\text{aff}}^{\otimes 2}$  *normally ordered*, if  $H(b \otimes b') > 0$ . For such a pair,  $G(b) \wedge G(b')$  is called a *normally ordered wedge*.

Note that  $v_1 \wedge v_1$  is normally ordered and is *not* equal to 0 even at  $q = 1$ .

The elements  $C_{b, b'} \in N$  ( $b, b' \in B_{\text{aff}}$  such that  $H(b \otimes b') = 0$ ) should be thought of as a rule for writing  $G(b) \wedge G(b')$  as a linear combination of normally ordered wedges.

**Proposition 3.2.**  $\{z^{a_1}v_{j_1} \wedge z^{a_2}v_{j_2}\}_{H(z^{a_1}b_{j_1} \otimes z^{a_2}b_{j_2}) > 0}$  is a base of  $\bigwedge^2 V_{\text{aff}}$ .

Let  $n \in \mathbb{Z}_{>0}$ . Define the  $n$ -wedge space by

$$N_n := \sum_{r=0}^{n-2} V_{\text{aff}}^{\otimes r} \otimes N \otimes V_{\text{aff}}^{\otimes (n-r-2)},$$

$$\bigwedge^n V_{\text{aff}} := V_{\text{aff}}^{\otimes n} / N_n.$$

By construction  $\bigwedge^n V_{\text{aff}}$  is a  $U_q(\mathfrak{g})$ -module.

A sequence  $(b_{j_1}, b_{j_2}, \dots, b_{j_n}) \in B_{\text{aff}}^{\otimes n}$  ( $j_r \in J$ ) is called *normally ordered* if  $H(b_{j_m} \otimes b_{j_{m+1}}) > 0$  for every  $m \in [0, n-1]$ . In this case the corresponding wedge  $v_{j_1} \wedge v_{j_2} \wedge \dots \wedge v_{j_n}$  is called a *normally ordered wedge*.

The homomorphism

$$\bigwedge^{r_1} V_{\text{aff}} \otimes \bigwedge^{r_2} V_{\text{aff}} \rightarrow \bigwedge^{r_1+r_2} V_{\text{aff}}$$

$$u_1 \otimes u_2 \mapsto u_1 \wedge u_2$$

is  $U_q(\mathfrak{g})$ -linear.

**Theorem 3.3.** *Normally ordered  $n$ -wedges form a base of  $\bigwedge^n V_{\text{aff}}$ .*

Define  $L(\bigwedge^n V_{\text{aff}})$  to be the image of  $L(V_{\text{aff}}^{\otimes n})$  in  $\bigwedge^n V_{\text{aff}}$  and  $B(\bigwedge^n V_{\text{aff}}) := \{(b_{j_1}, b_{j_2}, \dots, b_{j_n}) \mid \text{normally ordered}\}$ .

## 4 Level 2 Fock space

### 4.1 Ground state sequence

Recall that  $B_{\text{aff}}$  is the affinization of the perfect level  $k = 2$  crystal  $B$ . Extend the maps  $\varepsilon, \varphi : B \rightarrow P_{\text{cl}}$  to  $B_{\text{aff}} \rightarrow P_{\text{cl}}$  by  $\varepsilon(z^a b_j) := \varepsilon(b_j)$  and  $\varphi(z^a b_j) = \varphi(b_j)$ . Fix a sequence  $b_m^\circ \in B_{\text{aff}}$  ( $m \in \mathbb{Z}$ ) such that

$$\langle c, \varepsilon(b_m^\circ) \rangle = k,$$

$$\varepsilon(b_m^\circ) = \varphi(b_{m+1}^\circ),$$

$$H(b_m^\circ \otimes b_{m+1}^\circ) = 1.$$

We call this sequence a *ground state sequence*. Take also a sequence of weights  $\lambda_m$  ( $m \in \mathbb{Z}$ ) such that

$$\langle c, \lambda_m \rangle = k,$$

$$\text{cl}(\lambda_m) = \varphi(b_m^\circ),$$

$$\lambda_m = \text{wt}(b_m^\circ) + \lambda_{m+1}.$$

Define  $v_m^\circ := G(b_m^\circ)$ .

For our example of level 2  $U_q(\hat{\mathfrak{sl}}_2)$  there are, up to equivalence, only two possible ground state sequences

$$\begin{aligned} b_m^\circ &= \begin{cases} zb_2 & \text{for } m \in 2\mathbb{Z}, \\ b_0 & \text{for } m \in 2\mathbb{Z} + 1, \end{cases} \\ \lambda_m &= \begin{cases} 2\Lambda_0 + \delta & \text{for } m \in 2\mathbb{Z}, \\ 2\Lambda_1 & \text{for } m \in 2\mathbb{Z} + 1, \end{cases} \end{aligned} \quad (\text{A})$$

and

$$\begin{aligned} b_m^\circ &= b_1 & (m \in \mathbb{Z}), \\ \lambda_m &= \Lambda_0 + \Lambda_1 & (m \in \mathbb{Z}). \end{aligned} \quad (\text{B})$$

## 4.2 Fock space

In the formal semi-infinite wedge space  $\bigwedge^\infty V_{\text{aff}}$ , define the vacuum vector

$$\overline{|m\rangle} := v_m^\circ \wedge v_{m+1}^\circ \wedge v_{m+2}^\circ \wedge \cdots \quad (m \in \mathbb{Z}).$$

Define the pre-Fock space

$$\bar{\mathcal{F}}_m := \sum_{r \in \mathbb{N}} \bigwedge^r V_{\text{aff}} \wedge \overline{|m+r\rangle}.$$

Let  $L(\bar{\mathcal{F}}_m)$  be the crystal lattice of  $\bar{\mathcal{F}}_m$ . Define the Fock space  $\mathcal{F}_m$  to be

$$\mathcal{F}_m := \bar{\mathcal{F}}_m / \bigcap_{n>0} q^n L(\bar{\mathcal{F}}_m).$$

Let  $|m\rangle$  be the image of  $\overline{|m\rangle}$  in  $\mathcal{F}_m$ . Taking  $\{q^n L(\mathcal{F}_m)\}_{n \in \mathbb{N}}$  as a neighborhood system of 0,  $\mathcal{F}_m$  is endowed with a  $q$ -adic topology. By construction the  $q$ -adic topology is separated, since  $\bigcap_{n>0} q^n L(\mathcal{F}_m) = 0$ .

We have an algebra homomorphism  $\bigwedge^r V_{\text{aff}} \otimes \mathcal{F}_m \rightarrow \mathcal{F}_{m-r}$ .

Denote the vacuum vector  $|m\rangle$  associated to ground sequences (A) and (B) by  $|m\rangle^A$  and  $|m\rangle^B$  respectively. Similarly denote the Fock space  $\mathcal{F}_m$  associated to ground sequences (A) and (B) by  $\mathcal{F}_m^A$  and  $\mathcal{F}_m^B$  respectively.

Note that  $G(b) \wedge \overline{|m\rangle} = 0$ , if and only if there exists  $r \in \mathbb{N}$  such that  $G(b) \wedge v_m^\circ \wedge v_{m+1}^\circ \wedge \cdots \wedge v_{m+r}^\circ = 0$ .

**Theorem 4.1.** *Let  $b \in B_{\text{aff}}$  be such that  $H(b \otimes b_m^\circ) \leq 0$ . In  $\mathcal{F}_m$  the equality*

$$G(b) \wedge |m\rangle = 0$$

*holds in the  $q$ -adic topology.*



Note that this theorem only holds for a lower global base.

For example in  $\mathcal{F}_m^B$  the vacuum vector is  $|m\rangle^B = v_1 \wedge v_1 \wedge v_1 \wedge \dots$  and

$$\begin{aligned} v_0 \wedge |m\rangle^B &= \lim_{r \rightarrow \infty} (-q^2)^r v_1^{\wedge r} \wedge v_0 \wedge |m+r\rangle^B \\ &= 0 \quad (\text{in the } q\text{-adic topology}). \end{aligned} \quad (4.2.1)$$

For a normally ordered sequence  $(z^{a_m} b_{j_m}, z^{a_{m+1}} b_{j_{m+1}}, \dots)$  in  $B_{\text{aff}}$  such that  $z^{a_r} b_{j_r} = b_r^\circ$  for all  $r \gg m$ , the wedge  $z^{a_m} v_{j_m} \wedge z^{a_{m+1}} v_{j_{m+1}} \wedge \dots$  is called a *normally ordered wedge*.

**Theorem 4.2.** *The normally ordered wedges in  $\mathcal{F}_m$  form a base of  $\mathcal{F}_m$ .*

### 4.3 $U_q(\mathfrak{g})$ -module structure

First we assign weights to the Fock space by setting

$$\text{wt}(|m\rangle) := \lambda_m.$$

This fixes the action of the  $q^h$  ( $h \in P^*$ ). For example  $\text{wt}(|m\rangle^B) = \Lambda_0 + \Lambda_1$  ( $m \in \mathbb{Z}$ ).

**Proposition 4.3.** *Let  $V(\lambda_m)$  denote the irreducible integrable  $U_q(\mathfrak{g})$ -module of highest weight  $\lambda_m$ . The character of  $\mathcal{F}_m$  is*

$$\text{ch}(\mathcal{F}_m) = \text{ch}(V(\lambda_m)) \prod_{r>0} (1 - e^{-r\delta})^{-1}. \quad (4.3.1)$$

**Theorem 4.4.**  *$\mathcal{F}_m$  has the structure of an integrable  $U_q(\mathfrak{g})$ -module.*

There are two steps to proving this: (i) proving that the action of  $e_i, f_i$  is well-defined (converges in the  $q$ -adic topology) and is integrable, and (ii) checking that the commutation relations are satisfied. In fact it is sufficient to prove it on the vacuum vector  $|m\rangle$ .

For example in  $\mathcal{F}_m^B$  we have

$$\begin{aligned} t_i |m\rangle^B &= q |m\rangle^B \quad (i \in I), \\ e_1 |m\rangle^B &= [2] \sum_{r \in \mathbb{N}} v_1^{\wedge r} \wedge v_0 \wedge |m+1+r\rangle^B = 0 \quad (\text{by (4.2.1)}), \\ f_1 |m\rangle &= [2] \sum_{r \in \mathbb{N}} v_1^{\wedge r} \wedge v_2 \wedge q |m+1+r\rangle^B \\ &= q [2] \left( \sum_{r \in \mathbb{N}} (-q^2)^r \right) v_2 \wedge |m+1\rangle^B = v_2 \wedge |m+1\rangle^B. \end{aligned}$$

Then one can check that

$$\begin{aligned} [e_1, f_1] |m\rangle^B &= e_1 \cdot f_1 |m\rangle^B = e_1 v_2 \wedge |m+1\rangle^B \\ &= |m\rangle^B = \frac{t_1 - t_1^{-1}}{q - q^{-1}} |m\rangle^B. \end{aligned}$$

#### 4.4 Bosons

Define boson operators  $B_a$

$$\begin{aligned} B_a &:= \sum_{r \in \mathbb{N}} 1^{\otimes r} \otimes z^a \otimes 1^{\otimes \infty} \quad (a \in \mathbb{Z} \setminus \{0\}) \\ &\equiv z^a \otimes 1^{\otimes \infty} + 1 \otimes z^a \otimes 1^{\otimes \infty} + 1 \otimes 1 \otimes z^a \otimes 1^{\otimes \infty} + \dots \end{aligned}$$

**Proposition 4.5.** *The action of the operators  $B_a$  on  $\mathcal{F}_m$  converges in the  $q$ -adic topology.*

**Proposition 4.6.**  $B_a |m\rangle = 0$  for all  $a \in \mathbb{Z}_{>0}$ .

**Proposition 4.7.** *There exists  $\gamma_a \in \mathbb{Q}(q)$  (independent of  $m$ ) such that*

$$[B_a, B_{a'}] |m\rangle = \gamma_a \delta_{a+a',0} |m\rangle.$$

At  $q = 0$ ,  $\gamma_a = a$ .

Let  $H$  be the Heisenberg algebra generated by  $\{B_a\}_{a \in \mathbb{Z} \setminus \{0\}}$  with the defining relations  $[B_a, B_{a'}] = \delta_{a+a',0} \gamma_a$ . Then  $H$  acts on the Fock space  $\mathcal{F}_m$  commuting with the action of  $U_q(\mathfrak{g}')$ . Let  $\mathbb{Q}[H_-] := \mathbb{Q}[B_{-a}]_{a \in \mathbb{Z}_{>0}} \cdot 1$  be the Fock space for  $H$  with vacuum vector 1 and the defining relation  $B_a \cdot 1 = 0$  ( $a \in \mathbb{Z}_{>0}$ ). Let  $u_{\lambda_m}$  denote the highest weight vector in  $V(\lambda_m)$ . Since  $|m\rangle$  is annihilated by  $e_i$  ( $i \in I$ ) and  $B_a$  ( $a \in \mathbb{Z}_{>0}$ ), we have an injective  $U_q(\mathfrak{g}') \otimes H$ -linear homomorphism

$$\begin{aligned} \iota_m : V(\lambda_m) \otimes \mathbb{Q}[H_-] &\rightarrow \mathcal{F}_m \\ u_{\lambda_m} \otimes 1 &\mapsto |m\rangle. \end{aligned}$$

**Theorem 4.8.**  $\iota_m : V(\lambda_m) \otimes \mathbb{Q}[H_-] \rightarrow \mathcal{F}_m$  is an isomorphism.

The proof is by comparing the characters (4.3.1). This gives the decomposition of  $\mathcal{F}_m$  into irreducible  $U_q(\mathfrak{g})$ -modules.

$\gamma_a$  can be calculated by using the decomposition via  $\iota_m$  of the wedge vertex operator ( $V_{\text{aff}} \otimes \mathcal{F}_m \rightarrow \mathcal{F}_{m-1}$ ) to a product of the usual vertex operator ( $V_{\text{aff}} \otimes V(\lambda_m) \rightarrow V(\lambda_{m-1})$ ) and the boson vertex operator, and then calculating the equality of two-point functions corresponding to this decomposition (see [KMPY]).

**Proposition 4.9.** *In our example*

$$[B_a, B_{a'}] |m\rangle^B = \frac{a}{1 - q^{2a}} \delta_{a+a',0} |m\rangle^B.$$

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R.I.M.S., Kyoto University, Kyoto 606-01, Japan  
petersen@kurims.kyoto-u.ac.jp  
<http://www.kurims.kyoto-u.ac.jp/~petersen/>